

NOTE: ANSWER FIVE QUESTIONS ONLY

Q1. A) For what value of k is

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} + k & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

Continuous at every x ?

N
(GS)
72

✓
10marks

B) Find the equation of the plane having the two points A(2,1,1), B(7,5,2) and that is normal to the plane whose equation is $2x - 3x - 5z = 4$. 10marks

Q2. A) Find $\frac{dy}{dx}$ if:

1) $f(x) = \frac{\sin^2(2x+1)}{\cos^3\sqrt{x^2+x}}$

2) $f(x) = (x^2 + 1)^\pi + \pi^{\sin x}$

10marks

B) Evaluate 1) $\int \sin x \cos(\cos x + 2) dx$, 2) $\int \frac{\sin^3(\frac{1}{x^2}) \cos(\frac{1}{x^2})}{x^3} dx$

10marks

Q3. A) Compute the arc length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \leq x \leq 4$.

15marks

B) 1) Calculate the derivative $f(x) = \int_{\sqrt{x}}^{x^2} \frac{dt}{1+t+\sin t}$

5marks

2) Find $\frac{d}{dx} \left(\int_{\cos x}^{x^2} \frac{dt}{1+t+\sin t} \right)$

5marks

Q4. A) Approximate $(8.2)^{-\frac{1}{3}}$.

10marks

B) Evaluate the integral: 1) $\int x^6 \ln x dx$, 2) $\int \frac{3x^3 + 3x^2 - 12x + 20}{x^4 + 4x^2} dx$. 10marks

Q5. A) Evaluate: 1) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$, 2) $\int x \sqrt{1-x} dx$

10marks

B) Find the area of the region between the graphs of $f(x) = x^2$ and $g(x) = \sqrt{8x}$ for $1 \leq x \leq 3$. 10marks

Q6. A) Find the area of ABC if A(1,0,2), B(5,1,6) and C(3,3,1).

10marks

B) Evaluate the limit:

1) $\lim_{x \rightarrow 0} \frac{x^3 \sin \frac{1}{x}}{\sin x^2}$

2) $\lim_{x \rightarrow 1} (x)^{\frac{1}{1-x}}$

10marks

✓

$$Q_1: A) f(x) = \begin{cases} \frac{x^2-1}{x-1} + k & , \text{if } x < 1 \\ x^2 & , \text{if } x \geq 1 \end{cases}$$

Solution: Note that f is continuous for all x if it is continuous at $x=1$.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \left(\frac{x^2-1}{x-1} + k \right) = \lim_{x \rightarrow 1^+} (x+1+k) = k+2$$

$$\text{and } \lim_{x \rightarrow 1^+} = \lim_{x \rightarrow 1^+} x^2 = 1$$

For $f(x)$ to be continuous we must have

$$k+2=1 \Rightarrow k=-1$$

$$B) A(2,1,1), B(7,5,2) \in M_1$$

$$\vec{AB} = 5i + 4j + k$$

$$\nabla_1 \perp M_2 \Rightarrow \vec{N}_1 = 2i - 3j - 5k$$

$$\vec{AB} \times \vec{N}_1 \perp M_1$$

$$\vec{AB} \times \vec{N} = \begin{vmatrix} i & j & k \\ 5 & 4 & 1 \\ 2 & -3 & -5 \end{vmatrix} = -17i + 27j - 23k$$

$$M_1: -17x + 27y - 23z = -30$$

$$Q_2 \text{ A) Find } \frac{dy}{dx} \text{ if } f(x) = \frac{\sin^2(2x+1)}{\cos^3 \sqrt{x^2+x}}$$

$$Q \quad f'(x) = \frac{(\cos^3 \sqrt{x^2+x})(4\sin(2x+1)\cos(2x+1) - \sin^2(2x+1)/3\cos^2 \sqrt{x^2+x})(-\sin \sqrt{x^2+x})}{\cos^6 \sqrt{x^2+x}}$$

$\frac{2x+1}{2\sqrt{x^2+x}}$

$$\textcircled{2} \quad f(x) = (x^2+1)^{\pi} + \pi^{\sin x}$$

$$f'(x) = \pi(x^2+1)^{\pi-1} \cdot 2x + (\pi^{\sin x})(\cos x) \text{ by P.T}$$

$$B) \quad \textcircled{1} \int \sin x \cos(\cos x + 2) dx = -\sin(\cos x + 2) + C$$

$$\textcircled{2} \quad \int \frac{\sin^3(\frac{1}{x^2}) \cos \frac{1}{x^2}}{x^3} dx = \frac{-1}{2} \frac{\sin^4 \frac{1}{x^2}}{4} + C$$

$$Q_3 \text{ A) } y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$$

$$y' = \frac{1}{2}x - \frac{1}{2x}$$

$$1+(y')^2 = 1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} \\ = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$$

$$L = \int \sqrt{1+(y')^2} dx = \int \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2} dx = \int \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln x \right]_1^4 = \frac{15}{4} + \ln 2$$

$$Q_2 \text{ A) Find } \frac{dy}{dx} \text{ if } f(x) = \frac{\sin^2(2x+1)}{\cos^3 \sqrt{x^2+x}}$$

$$Q \quad f'(x) = \frac{(\cos^3 \sqrt{x^2+x})(4\sin(2x+1)\cos(2x+1) - \sin^2(2x+1)/3\cos^2 \sqrt{x^2+x})(-\sin \sqrt{x^2+x})}{\cos^6 \sqrt{x^2+x}} \cdot \frac{2x+1}{2\sqrt{x^2+x}}$$

$$\textcircled{2} \quad f(x) = (x^2+1)^{\pi} + \pi^{\sin x}$$

$$f'(x) = \pi(x^2+1)^{\pi-1} (2x) + (\pi^{\sin x})(\cos x) \text{ by P.T}$$

$$B) \quad \textcircled{1} \int \sin x \cos(\cos x + 2) dx = -\sin(\cos x + 2) + C$$

$$\textcircled{2} \quad \int \frac{\sin^3(\frac{1}{x^2}) \cos \frac{1}{x^2}}{x^3} dx = \frac{-1}{2} \frac{\sin^4 \frac{1}{x^2}}{4} + C$$

$$Q_3 \text{ A) } y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$$

$$y' = \frac{1}{2}x - \frac{1}{2x}$$

$$1+(y')^2 = 1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} \\ = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$$

$$L = \int \sqrt{1+(y')^2} dx = \int \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2} dx = \int \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln x \right]_1^4 = \frac{15}{4} + \ln 2$$

$$B) f'(x) = \frac{2x}{1+x^2+\sin x^2} - \frac{1}{1+\sqrt{x}+\sin \sqrt{x}}$$

$$Q_4 A) f(x) = x^{-\frac{1}{3}} \text{ so } f'(x) = -\frac{1}{3} x^{-\frac{4}{3}}$$

$x_0 = 8$

$$f(8.2) = f(8) + \Delta y$$

$$\Delta y \approx dy = f'(8) dx$$

$$= -\frac{1}{3}(8)(8.2-8) = \frac{1}{240}$$

$$f(8.2) = f(8) + dy = \frac{1}{2} - \frac{1}{240} \approx 0.495$$

$$B) \quad ① \int x^6 \ln x \, dx$$

$$u = \ln x \quad & du = x^6 dx$$

$$du = \frac{1}{x} dx, \quad v = \frac{x^7}{7}$$

$$\int u \, dv = uv - \int v \, du - \cancel{\int u \, dv} -$$

$$= \frac{x^7}{7} \ln x - \int \frac{x^7}{7} \frac{1}{x} dx = \frac{x^7}{7} \ln x - \frac{x^7}{7} + C$$

$$② \int \frac{3x^3 + 3x^2 - 12x + 20}{x^4 + 4x^2} dx$$

$$\frac{3x^3 + 3x^2 - 12x + 20}{x^2(x^2 + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2}$$

$$3x^3 + 3x^2 - 12x + 20 = A \cdot x(x^2 + 4) + B(x^2 + 4) + (Cx + D)x^2$$

$$= (Ax^3 + Cx^2) + (Bx^2 + Dx^2) + 4Ax + 4B$$

$$\text{which implies } A + C = 3, B + D = 3, 4A = -12, 4B = 20$$

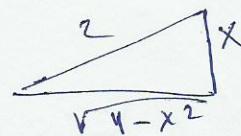
and we have

$$\begin{aligned} \int \frac{3x^3 + 3x^2 - 12x + 20}{x^2(x^2+4)} dx &= \int \left(\frac{-3}{x} + \frac{5}{x^2} + \frac{6x-2}{x^2+4} \right) dx \\ &= -3 \int \frac{1}{x} dx + 5 \int \frac{1}{x^2} dx + 3 \int \frac{2x}{x^2+4} dx - \frac{1}{4} \int \frac{1}{x^2+1} dx \\ &= -3 \ln|x| - \frac{5}{x} + 3 \ln(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right) + C. \end{aligned}$$

Q5 A) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$

Let $x = 2 \sin \theta$

so that $dx = 2 \cos \theta d\theta$



$$\begin{aligned} \int \frac{x^2}{\sqrt{4-x^2}} dx &= \int \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} 2 \cos \theta d\theta \\ &= \int 4 \sin^2 \theta d\theta = 2 \int (1 - \cos 2\theta) d\theta \\ &= 2 \left[\theta - \frac{\sin 2\theta}{2} \right] \\ &= 2 \left[\sin^{-1} \frac{x}{2} - \frac{2 \sin \theta \cos \theta}{2} \right] \\ &= \left[2 \sin^{-1} \frac{x}{2} - 2 \frac{x}{2} \frac{\sqrt{4-x^2}}{2} \right]_0^1 \end{aligned}$$

$$\begin{aligned} &= \left(2 \sin^{-1} 1 - \frac{\sqrt{3}}{2} \right) - (0-0) \\ &= 2 \sin^{-1} 1 - \frac{\sqrt{3}}{2} \end{aligned}$$

② $\int x \sqrt{1-x} dx$ $u = 1-x \rightarrow du = -dx$

$$\begin{aligned} &= \int (1-u) \sqrt{u} (-du) = \int (-\sqrt{u} + u^{3/2}) du \\ &= -\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2} + C = -\frac{(1-x)^{3/2}}{3/2} - \frac{(1-x)^{5/2}}{5/2} + C \end{aligned}$$

3/1

B) that since the two curves
in the middle of the interval,

$$\sqrt{8x} = x^2, \text{ so that}$$

$$8x = x^4 \text{ or } x(x^3 - 8) = 0$$

$$\text{or } x=0, x=2, \text{ since } x=0 \notin [1, 3]$$

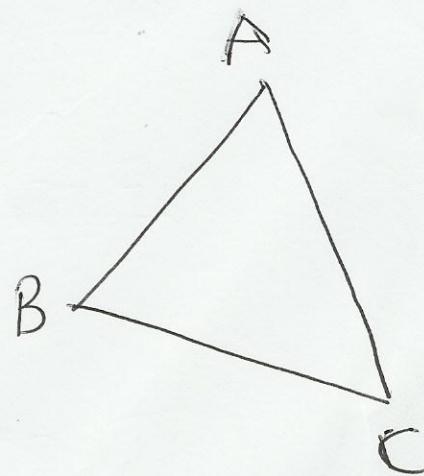
$$\begin{aligned}\text{Area} &= \int_1^2 [\sqrt{8x} - x^2] dx + \int_2^3 [x^2 - \sqrt{8x}] dx \\ &= \left[\sqrt{8} \frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_1^2 + \left[\frac{x^3}{3} - \sqrt{8} \frac{x^{3/2}}{3/2} \right]_2^3 \\ &\quad + \left[\frac{3^3}{3} - \frac{2\sqrt{8}}{3} 3^{3/2} \right] - \left[\frac{2^3}{3} - \frac{2\sqrt{8}}{3} 2^{3/2} \right] \\ &= \left(\frac{16}{3} - \frac{8}{3} \right) - \left(\frac{4\sqrt{8}}{3} - \frac{1}{3} \right) + \left(\frac{27}{3} - 4\sqrt{8} \right) - \left(\frac{8}{3} - \frac{16}{3} \right) \\ &= \frac{4 - 4\sqrt{8} - 12\sqrt{6}}{3}\end{aligned}$$

Q) A(1, 0, 2), B(5, 1, 6) and C(3, 3, 1).

$$\vec{AB} = 5i - j + 4k$$

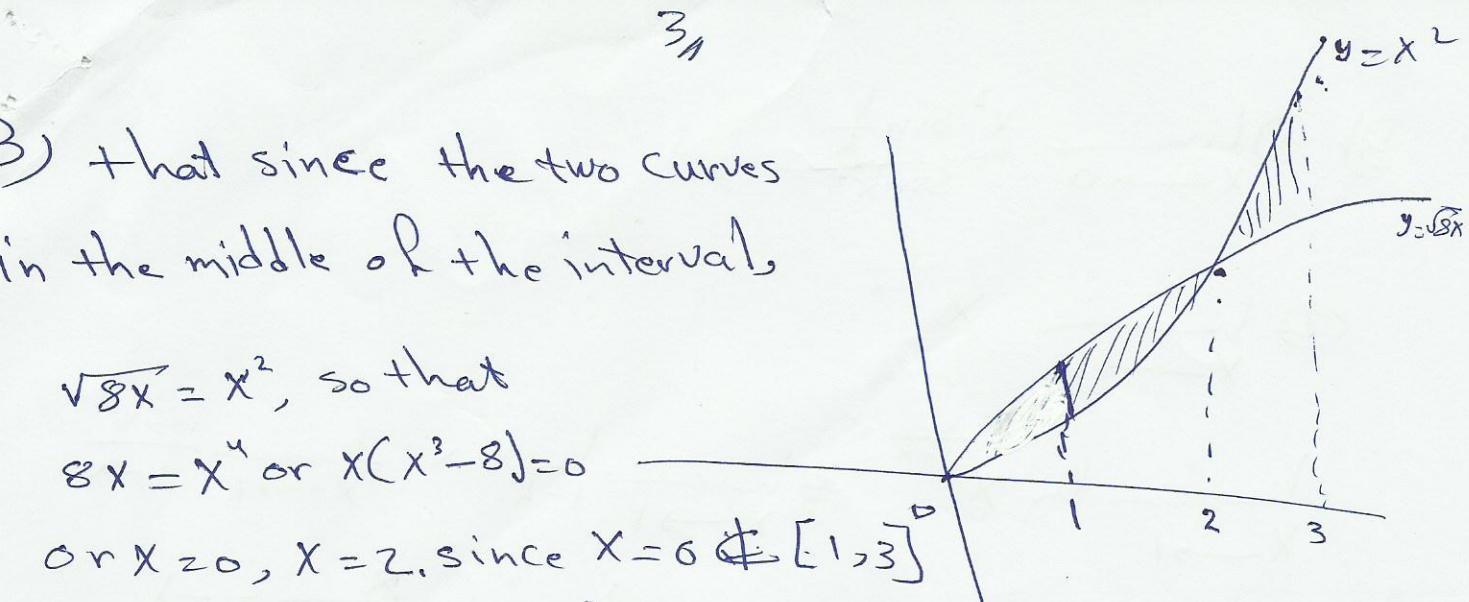
$$\vec{BC} = -2i + 2j - 5k$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 5 & -1 & 4 \\ -2 & 2 & -5 \end{vmatrix}$$



$$\begin{aligned}&= (5-8)i - (-25+8)j + (10-2)k \\&= -3i - (-18)j + 8k = -3i + 18j + 8k\end{aligned}$$

$$|\vec{AB} \times \vec{BC}| = \sqrt{9 + (18)^2 + 8^2} = \boxed{\text{_____}}$$



B) ① $\lim_{x \rightarrow 0} \frac{x^3 \sin \frac{1}{x}}{\sin x^2} = 1$

② $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$

$$\lim_{x \rightarrow 1} \ln x^{\frac{1}{1-x}} = \lim_{x \rightarrow 1} \frac{1}{1-x} \ln x = \lim_{x \rightarrow 1} \frac{\ln x}{1-x}$$

$$= \lim_{x \rightarrow 1} \frac{\cancel{\infty}}{-1} = -1$$

$$\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = e^{-1}$$